Exercise 05 CMM. 15.05.2024

Displacement Formulation for Linear Elastic Wave Propagation.
- Discussed in the Lecture.

$\Rightarrow$ Velocity - Stress Formulation

$\underline{H} \Rightarrow$ Displacement Gradient

$$\rho \frac{\partial^2 \underline{w}}{\partial t^2} = \nabla \cdot \underline{\underline{\sigma}}$$

Balance of Linear Momentum

$$\boxed{\rho \frac{\partial \underline{v}}{\partial t} = \nabla \cdot \underline{\underline{\sigma}}}$$

$$\underline{\underline{D}} = \frac{1}{2}\left(\underline{\underline{H}} + \underline{\underline{H}}^T\right)$$

Constitutive Relations

$$\boxed{\underline{\underline{\sigma}} = 2\mu \underline{\underline{D}} + \lambda \, tr(\underline{\underline{D}}) \underline{\underline{I}}}$$

Linear Elasticity

$$\underline{v} = \frac{\partial w}{\partial t}$$

$\rightarrow$ 2-D Case : x, y

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \cancel{2}\mu \begin{bmatrix} 2\frac{\partial w_x}{\partial x} & \frac{\partial w_y}{\partial x} + \frac{\partial w_x}{\partial y} \\ sym & 2\frac{\partial w_y}{\partial y} \end{bmatrix} + \lambda \begin{bmatrix} \cancel{2}\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right) & 0 \\ 0 & \cancel{2}\left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right) \end{bmatrix}$$

x MB: $$\frac{\partial V_x}{\partial t} - \frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x} - \frac{1}{\rho}\frac{\partial \sigma_{xy}}{\partial y} = 0 \quad -①$$

y MB: $$\frac{\partial V_y}{\partial t} - \frac{1}{\rho}\frac{\partial_{xy}}{\partial x} - \frac{1}{\rho}\frac{\partial \sigma_{yy}}{\partial y} = 0 \quad -②$$

$$\frac{\partial}{\partial t}\left[\sigma_{xx} = 2\mu \frac{\partial w_x}{\partial x} + \lambda \frac{\partial w_x}{\partial x} + \lambda \frac{\partial w_y}{\partial y}\right] \quad -③$$

$$\frac{\partial}{\partial t}\left[\sigma_{yy} = 2\mu \frac{\partial w_y}{\partial y} + \lambda \frac{\partial w_y}{\partial y} + \lambda \frac{\partial w_x}{\partial x}\right] \quad -④$$

$$\frac{\partial}{\partial t}\left[\sigma_{xy} = \mu \frac{\partial w_y}{\partial x} + \mu \frac{\partial w_x}{\partial y}\right] \quad -⑤$$

Let

$$\underline{Q} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ V_x \\ V_y \end{bmatrix} \quad -⑥$$

Combining eq ①,②,③,④,⑤ in matrix form using ⑥ we get

$$\frac{\partial Q}{\partial t} - \underline{\underline{A}}\frac{\partial Q}{\partial x} - \underline{\underline{B}}\frac{\partial Q}{\partial y} = 0 \longrightarrow \text{Velocity stress formulation}$$

where,

$$A = \begin{bmatrix} 0 & 0 & 0 & \lambda+2\mu & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \mu \\ \rho^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^{-1} & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & \lambda+2\mu \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & \rho^{-1} & 0 & 0 \\ 0 & \rho^{-1} & 0 & 0 & 0 \end{bmatrix}$$

Wave Equation in y direction

$$Q(x,y,t) = Q_0 \exp(i(ky - \omega t))$$

$$\frac{\partial Q}{\partial x} = 0 \qquad \frac{\partial Q}{\partial y} = iQ_0 k \exp(i(ky-\omega t)) = ik\underline{Q}$$

$$\frac{\partial Q}{\partial t} = -\omega i Q$$

$$-\omega i\underline{Q} - \underline{\underline{A}}(0) - \underline{\underline{B}}\, ik\underline{Q} = 0$$

$$(-\omega i\underline{I} - Bik)\, i\underline{Q} = 0$$

$$(-\omega I - kB)\, i\underline{Q} = 0 \Longrightarrow \det(\omega I + kB) = 0$$

$$\det\begin{vmatrix} \omega & 0 & 0 & 0 & k\lambda \\ 0 & \omega & 0 & 0 & k(\lambda+2\mu) \\ 0 & 0 & \omega & k\mu & 0 \\ 0 & 0 & k\rho^{-1} & \omega & 0 \\ 0 & k\rho^{-1} & 0 & 0 & \omega \end{vmatrix} = 0$$

$$\det(\omega \underline{\underline{I}} + k \underline{\underline{B}}) = \omega^5 - \omega^3 k^2 \mu \rho^{-1} - \omega^3 k^2 \rho^{-1}(\lambda+2\mu) + \omega k^4 \rho^{-2}(\lambda+2\mu)\mu$$

$\omega_1 = 0$ OR $\omega^4 - \omega^2 k^2 \rho^{-1}(\lambda+3\mu) + k^4 \rho^{-2}(\lambda+2\mu)\mu = 0$

$\omega^2 = a$
$k^2 = b$

$$\omega^2 = \frac{1}{2} k^2 \rho^{-1}(\lambda + 3\mu \pm (\lambda+\mu))$$

$$\omega_2 = k\sqrt{\frac{\lambda+2\mu}{\rho}}, \quad \omega_3 = k\sqrt{\frac{\mu}{\rho}}, \quad \omega_4 = -k\sqrt{\frac{\lambda+2\mu}{\rho}}, \quad \omega_5 = -k\sqrt{\frac{\mu}{\rho}}$$

$\Rightarrow$ Wave speed $c = \dfrac{\omega}{k}$

$$c_2 = \sqrt{\frac{\lambda+2\mu}{\rho}}, \quad c_3 = \sqrt{\frac{\mu}{\rho}}, \quad c_4 = -\sqrt{\frac{\lambda+2\mu}{\rho}}, \quad c_5 = -\sqrt{\frac{\mu}{\rho}}$$

HW: $$\frac{\lambda+2\mu}{\rho} = \frac{\dfrac{\nu E}{(1+\nu)(1-2\nu)} + \dfrac{2E}{2(1+\nu)}}{\rho}$$

$$= \frac{2(1-\nu)}{1-2\nu}\frac{\mu}{\rho}$$

Two waves in +ve & -ve direction with wave speeds $\sqrt{\dfrac{\lambda+2\mu}{\rho}}$ & $\sqrt{\dfrac{\mu}{\rho}}$

Problem : Linearization & Small Deformation (Displacements)

$$\underline{\underline{Q}}\,\underline{\underline{U}} = \underline{\underline{F}} = \underline{\underline{I}} + \underline{\underline{H}}$$

$$F = \begin{bmatrix} 1.002 & 0.001 & 0 \\ 0 & 0.998 & 0 \\ 0.002 & 0 & 0.999 \end{bmatrix}$$

F : Deformation Gradient

I : Identity Tensor

H : Displacement Gradient.

$\underline{\underline{H}} = ?$ $\quad \underline{\underline{U}} = ?$ $\quad \underline{\underline{B}} = ?$ $\quad \underline{\underline{Q}} = ?$

$\underline{\underline{U}}_{lin} = ?$ $\quad \underline{\underline{B}}_{lin} = ?$ $\quad \underline{\underline{Q}}_{lin} = ?$

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}\right) \qquad \underline{\underline{D}} = \frac{1}{2}\left(\underline{\underline{H}} + \underline{\underline{H}}^T\right)$$

$$B = \underline{\underline{F}}\,\underline{\underline{F}}^T \qquad U = (FF^T)^{\frac{1}{2}} \qquad Q = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$$

$$U_{lin} = \underline{\underline{I}} + \frac{1}{2}\left(\underline{\underline{H}} + \underline{\underline{H}}^T\right), \quad B_{lin} = \underline{\underline{I}} + \underline{\underline{H}} + \underline{\underline{H}}^T, \quad \underline{\underline{Q}}_{lin} = \underline{\underline{I}} + \frac{1}{2}\left(\underline{\underline{H}} - \underline{\underline{H}}^T\right)$$